arXiv:1508.04203v2 [math.AP] 12 Oct 2015

Convergence Rates in Homogenization of Stokes Systems

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Abstract

This paper studies the convergence rates in L^2 and H^1 of Dirichelt problems for Stokes systems with rapidly oscillating periodic coefficients, without any regularity assumptions on the coefficients.

Keywords: Convergence rates; Stokes systems; Homogenization.

1 Introduction and Main Results

The purpose of this paper is to study the convergence rates of Dirichlet problems for Stokes systems with rapidly oscillating periodic coefficients. More precisely, we consider the following Dirichlet problem for Stokes systems associated with matrix A,

$$\begin{cases} \mathcal{L}_\varepsilon(u_\varepsilon) + \nabla p_\varepsilon = F & \text{in } \Omega, \\ \text{div } u_\varepsilon = g & \text{in } \Omega, \\ u_\varepsilon = f & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

with the compatibility condition

$$\int_\Omega g - \int_{\partial\Omega} f \cdot n = 0, \tag{1.2}$$

where n denotes the outward unit normal to $\partial\Omega$ and $\Omega \subset \mathbb{R}^d$ is a bounded domain. We note that the Dirichlet problem (1.1) is used in the modeling of flows in porous media. Here $\varepsilon > 0$ is a small parameter and the operator $\mathcal{L}_\varepsilon$ is defined by

$$\mathcal{L}_\varepsilon = -\text{div}(A(x/\varepsilon)\nabla) = -\frac{\partial}{\partial x_i}\left[a_{ij}^{\alpha\beta}\left(\frac{x}{\varepsilon}\right)\frac{\partial}{\partial x_j}\right] \tag{1.3}$$

*Supported in part by NSF grant DMS-1161154

with $1 \le i, j, \alpha, \beta \le d$ (the summation convention is used throughout). We will assume that the coefficient matrix $A(y) = (a_{ij}^{\alpha\beta}(y))$ is real, bounded measurable, and satisfies the ellipticity condition:

$$\mu|\xi|^2 \le a_{ij}^{\alpha\beta}(y)\xi_i^\alpha \xi_j^\beta \le \frac{1}{\mu}|\xi|^2 \qquad \text{for } y \in \mathbb{R}^d \text{ and } \xi = (\xi_i^\alpha) \in \mathbb{R}^{d\times d}, \tag{1.4}$$

where $\mu > 0$. We also assume that $A(y)$ satisfies the periodicity condition,

$$A(y+z) = A(y) \qquad \text{for } y \in \mathbb{R}^d \text{ and } z \in \mathbb{Z}^d. \tag{1.5}$$

No symmetry condition on $A(y)$ is needed. A function satisfying (1.5) will be called 1-periodic.

By the homogenization theory of Stokes systems (see [2,6]), under suitable conditions on F, f and g, it is known that

$$u_\varepsilon \rightharpoonup u_0 \quad \text{weakly in } H^1(\Omega;\mathbb{R}^d) \quad \text{and} \quad p_\varepsilon - \fint_\Omega p_\varepsilon \rightharpoonup p_0 - \fint_\Omega p_0 \quad \text{weakly in } L^2(\Omega),$$

where $(u_0, p_0) \in H^1(\Omega;\mathbb{R}^d) \times L^2(\Omega)$ is the weak solution of the homogenized problem with constant coefficients,

$$\begin{cases} \mathcal{L}_0(u_0) + \nabla p_0 = F & \text{in } \Omega, \\ \operatorname{div} u_0 = g & \text{in } \Omega, \\ u_0 = f & \text{on } \partial\Omega. \end{cases} \tag{1.6}$$

The primary purpose of this paper is to investigate the rate of convergence of $\|u_\varepsilon - u_0\|_{L^2(\Omega)}$, as $\varepsilon \to 0$. The following is the main result of the paper.

Theorem 1.1. *Let Ω be a bounded $C^{1,1}$ domain. Suppose that A satisfies the ellipticity condition (1.4) and periodicity condition (1.5). Given $g \in H^1(\Omega)$ and $f \in H^{3/2}(\partial\Omega;\mathbb{R}^d)$ satisfying the compatibility condition(1.2), for $F \in L^2(\Omega;\mathbb{R}^d)$, let $(u_\varepsilon, p_\varepsilon)$, (u_0, p_0) be weak solutions of Dirichlet problems (1.1), (1.6), respectively. Then*

$$\|u_\varepsilon - u_0\|_{L^2(\Omega)} \le C\varepsilon \|u_0\|_{H^2(\Omega)}, \tag{1.7}$$

where the constant C depends only on d, μ, and Ω.

Theorem 1.1 gives the optimal $O(\varepsilon)$ convergence rate for the inverses of the Stokes operators in L^2 operator norm. Indeed, let $T_\varepsilon : F \in L^2_\sigma(\Omega) \to u_\varepsilon$, where $L^2_\sigma(\Omega) = \big\{F \in L^2(\Omega;\mathbb{R}^d) : \operatorname{div}(F) = 0 \text{ in } \Omega\big\}$, and u_ε denotes the solution of (1.1) with $F \in L^2_\sigma(\Omega;\mathbb{R}^d)$ and $g = 0$, $f = 0$. Then it follows from (1.7) and the estimate $\|u_0\|_{H^2(\Omega)} \le C\|F\|_{L^2(\Omega)}$ that

$$\|T_\varepsilon - T_0\|_{L^2_\sigma(\Omega)\to L^2_\sigma(\Omega)} \le C\varepsilon,$$

where $T_0 : F \in L^2_\sigma(\Omega) \to u_0$.

In this paper we also obtain $O(\sqrt{\varepsilon})$ rates for a two-scale expansion of $(u_\varepsilon, p_\varepsilon)$ in $H^1 \times L^2$. Let (χ, π) denote the correctors associated with A, defined by (2.5), and S_ε the Steklov smoothing operater defined by (2.1).

Theorem 1.2. *Let Ω be a bounded $C^{1,1}$ domain. Suppose that A satisfies (1.4) and (1.5). Let $(u_\varepsilon, p_\varepsilon)$ and (u_0, p_0) be the same as in Theorem 1.1. Then*

$$\|u_\varepsilon - u_0 - \varepsilon\chi^\varepsilon S_\varepsilon(\nabla\widetilde{u}_0)\|_{H^1(\Omega)} \le C\sqrt{\varepsilon}\|u_0\|_{H^2(\Omega)}, \tag{1.8}$$

where $\chi^\varepsilon(x) = \chi(x/\varepsilon)$ and $\widetilde{u}_0$ is the extension of u_0 defined as in (3.1). Moreover, if $\int_\Omega p_\varepsilon = \int_\Omega p_0 = 0$, then

$$\|p_\varepsilon - p_0 - \Big\{\pi^\varepsilon S_\varepsilon(\nabla\widetilde{u}_0) - \fint_\Omega \pi^\varepsilon S_\varepsilon(\nabla\widetilde{u}_0)\Big\}\|_{L^2(\Omega)} \le C\sqrt{\varepsilon}\|u_0\|_{H^2(\Omega)}, \tag{1.9}$$

where $\pi^\varepsilon(x) = \pi(x/\varepsilon)$. The constants C in (1.8) and (1.9) depend only on d, μ, and Ω.

We now describe the known L^2 convergence results on Dirichlet problems for general elliptic equations and systems with rapidly oscillating periodic coefficients. Consider the Dirichlet problem for the scalar elliptic equation $\mathcal{L}_\varepsilon(u_\varepsilon) = -\text{div}\big(A(x/\varepsilon)\nabla u_\varepsilon\big) = F$ in a Lipschitz domain Ω with $u_\varepsilon = f$ on $\partial\Omega$. It is well known that

$$\|u_\varepsilon - u_0\|_{L^2(\Omega)} \le C\varepsilon\left\{\|\nabla^2 u_0\|_{L^2(\Omega)} + \|\nabla u_0\|_{L^\infty(\partial\Omega)}\right\}. \tag{1.10}$$

To see (1.10), one considers the difference between u_ε and its first order approximation $u_0 + \varepsilon\chi^\varepsilon\nabla u_0$ and let

$$v_\varepsilon = u_\varepsilon - u_0 - \varepsilon\chi^\varepsilon\nabla u_0. \tag{1.11}$$

To correct the boundary data, one further introduces a function w_ε, where w_ε is the solution to the Dirichlet problem: $\mathcal{L}_\varepsilon(w_\varepsilon) = 0$ in Ω and $w_\varepsilon = -\varepsilon\chi^\varepsilon\nabla u_0$ on $\partial\Omega$. Using energy estimates, one may show that $\|v_\varepsilon - w_\varepsilon\|_{H_0^1(\Omega)} \le C\varepsilon\|\nabla^2 u_0\|_{L^2(\Omega)}$. The estimate (1.10) follows from this and the estimate $\|w_\varepsilon\|_{L^\infty(\Omega)} \le C\varepsilon\|\nabla u_0\|_{L^\infty(\partial\Omega)}$, which is obtained by the maximum principle (see e.g. [7]). More recently, Griso [4, 5] was able to establish the much sharper estimate (1.7), using the method of periodic unfolding. We mention that in the case of scalar elliptic equations with bounded measurable coefficients, one may also prove (1.7) by using the so-called Dirichlet corrector. In fact, it was shown in [9] that

$$\|u_\varepsilon - u_0 - \Big\{\Phi_\varepsilon - x\Big\}\nabla u_0\|_{H_0^1(\Omega)} \le C\varepsilon\|u_0\|_{H^2(\Omega)}, \tag{1.12}$$

where $\Phi_\varepsilon(x)$ is the solution of $\mathcal{L}_\varepsilon(\Phi_\varepsilon) = 0$ in Ω with $\Phi_\varepsilon = x$ on $\partial\Omega$. In the case of elliptic systems, the estimates (1.12) and thus (1.7) continue to hold under the additional assumption that A is Hölder continuous. Moreover, if A is Hölder continuous and symmetric, it was proved in [8] that

$$\|v_\varepsilon\|_{H^{1/2}(\Omega)} \le C\varepsilon\|u_0\|_{H^2(\Omega)}. \tag{1.13}$$

The approaches used in [8, 9] rely on the uniform regularity estimates established in [1, 10] and do not apply to operators with bounded measurable coefficients. Recently, by using the Steklov smoothing operator, T.A. Suslina [13, 14] was able to establish the $O(\varepsilon)$ estimate (1.7) in L^2 for a boarder class of elliptic operators, which, in particular, contains the elliptic systems $\mathcal{L}_\varepsilon$ in divergence form with coefficients satisfying the ellipticity condition $a_{ij}^{\alpha\beta}\xi_i^\alpha\xi_j^\beta \ge \mu|\xi|^2$ for any $\xi = \big(\xi_i^\alpha\big) \in \mathbb{R}^{m\times d}$. Since the correctors χ may not

be bounded in the case of nonsmooth coefficients, the idea is to consider the two-scale expansion

$$v_\varepsilon = u_\varepsilon - u_0 - \varepsilon \chi^\varepsilon S_\varepsilon(\nabla \widetilde{u}_0), \tag{1.14}$$

where S_ε is a smoothing operator at scale ε and $\widetilde{u}_0$ an extension of u_0 to $\mathbb{R}^d$ (also see [11, 12, 16] and their references on the use of S_ε in homogenization). This reduces the problem to the control of the L^2 norm of w_ε, where w_ε is the solution to the Dirichlet problem: $\mathcal{L}_\varepsilon(w_\varepsilon) = 0$ in Ω and $w_\varepsilon = -\varepsilon \chi^\varepsilon S_\varepsilon \nabla(\widetilde{u}_0)$ on $\partial\Omega$. Next, one considers

$$h_\varepsilon = w_\varepsilon - \varepsilon \chi^\varepsilon \theta_\varepsilon S_\varepsilon(\nabla \widetilde{u}_0),$$

where θ_ε is a cutoff function supported in an ε neighborhood of $\partial\Omega$. Note that $h_\varepsilon = 0$ on $\partial\Omega$ and $\mathcal{L}_\varepsilon(h_\varepsilon)$ is supported in an ε neighborhood of $\partial\Omega$. This allows one to approximate h_ε in the L^2 norm by h_0, using an $O(\sqrt{\varepsilon})$ estimate in H^1 and a duality argument, where $\mathcal{L}_0(h_0) = \mathcal{L}_\varepsilon(h_\varepsilon)$ in Ω and $h_0 = 0$ on $\partial\Omega$. Finally, one estimates the L^2 norm of h_0 by another duality argument.

In this paper we extend the approach of Suslina to the case of Stokes systems, which do not fit the standard framework of second-order elliptic systems in divergence form. As expected in the study of Stokes or Navies-Stokes systems, the main difficulty is caused by the pressure term p_ε. By carefully analyzing the systems for the correctors (χ, π) as well as their dual $(\phi_{kin}^{\alpha\beta}, q_{ij}^{\beta})$ (see Lemmas 3.1 and 3.3), we are able to establish the $O(\sqrt{\varepsilon})$ error estimates, given in Theorem 1.2, for the two-scale expansions of $(u_\varepsilon, p_\varepsilon)$ in $H^1 \times L^2$. This allows us to use the idea of boundary cutoff and duality argument in a manner similar to that in [13].

The paper is organized as follows. In Section 2 we recall a few basic properties of the Steklov smoothing operator S_ε as well as the homogenization theory for Stokes systems with periodic coefficients. In Section 3 we study $u_0 + \varepsilon \chi^\varepsilon S_\varepsilon \nabla \widetilde{u}_0$ as the first order approximation of u_ε. We introduce the dual correctors (Φ, q) and use energy estimates to establish the estimate (1.8) in H^1. In Section 4 we study the convergence of p_ε and prove the error estimate (1.9) for the two-scale expansion of the pressure term. Finally, our main theorem Theorem 1.1 is proved in Section 5. This is done by using the idea of boundary cutoff and duality, and by applying error estimates obtained in Sections 3 and 4 to the adjoint systems.

Throughout this paper, we denote $Y = [0, 1)^d$ and the L^1 average of f over the set E by

$$\fint_E f = \frac{1}{|E|} \int_E f.$$

We will use C to denote constants that may depend on d, μ, or Ω, but never on ε.
Acknowledgement. The author would like to thank referees for their very helpful comments and suggestions.

2 Preliminaries

2.1 Smoothing in Steklov's sense

Let S_ε be the operator on $L^2(\mathbb{R}^d)$ given by

$$(S_\varepsilon u)(x) = \fint_Y u(x - \varepsilon z)dz \tag{2.1}$$

and called the *Steklov smoothing operator*. Note that

$$\|S_\varepsilon u\|_{L^2(\mathbb{R}^d)} \le \|u\|_{L^2(\mathbb{R}^d)}.$$

Obviously, $D^\alpha S_\varepsilon u = S_\varepsilon D^\alpha u$ for $u \in H^s(\mathbb{R}^d)$ and any multi-index α such that $|\alpha| \le s$. Therefore,

$$\|S_\varepsilon u\|_{H^s(\mathbb{R}^d)} \le \|u\|_{H^s(\mathbb{R}^d)}.$$

The following are a few properties of Steklov's operator; see [13, 14].

Proposition 2.1. *For any $u \in H^1(\mathbb{R}^d)$ we have*

$$\|S_\varepsilon u - u\|_{L^2(\mathbb{R}^d)} \le C\varepsilon\|\nabla u\|_{L^2(\mathbb{R}^d)},$$

where C depends only on d.

We will use the notation $f^\varepsilon(x) = f(x/\varepsilon)$.

Proposition 2.2. *Let $f(x)$ be a 1-periodic function in $\mathbb{R}^d$ such that $f \in L^2(Y)$. Then for any $u \in L^2(\mathbb{R}^d)$,*

$$\|f^\varepsilon S_\varepsilon u\|_{L^2(\mathbb{R}^d)} \le \|f\|_{L^2(Y)}\|u\|_{L^2(\mathbb{R}^d)}.$$

2.2 Homogenization of Stokes systems

We refer the reader to [2, 6] for details of weak solutions and homogenization theory of Stokes system.

Let Ω be a bounded Lipschitz domain in $\mathbb{R}^d$. For $u, v \in H^1(\Omega; \mathbb{R}^d)$, we define the bilinear form $a_\varepsilon(\cdot, \cdot)$ by

$$a_\varepsilon(u, v) = \int_\Omega a_{ij}^{\alpha\beta}\left(\frac{x}{\varepsilon}\right)\frac{\partial u^\beta}{\partial x_j}\frac{\partial v^\alpha}{\partial x_i}dx.$$

For $F \in H^{-1}(\Omega; \mathbb{R}^d)$ and $g \in L^2(\Omega)$, we say that $(u_\varepsilon, p_\varepsilon) \in H^1(\Omega; \mathbb{R}^d) \times L^2(\Omega)$ is a weak solution of the following Stokes system in Ω,

$$\begin{cases} \mathcal{L}_\varepsilon(u_\varepsilon) + \nabla p_\varepsilon = F \\ \qquad\quad \operatorname{div} u_\varepsilon = g, \end{cases} \tag{2.2}$$

if for any $\varphi \in C_0^1(\Omega; \mathbb{R}^d)$,

$$a_\varepsilon(u_\varepsilon, \varphi) - \int_\Omega p_\varepsilon \operatorname{div}(\varphi) = \langle F, \varphi\rangle$$

and $\operatorname{div}(u_\varepsilon) = g$ in Ω (in the sense of distribution).

Theorem 2.3. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^d$. Suppose $A(y)$ satisfies the ellipticity condition (1.4). Let $F \in H^{-1}(\Omega; \mathbb{R}^d)$, $g \in L^2(\Omega)$ and $f \in H^{1/2}(\partial\Omega; \mathbb{R}^d)$ satisfy the compatibility condition (1.2). Then there exist a unique $u_\varepsilon \in H^1(\Omega; \mathbb{R}^d)$ and $p_\varepsilon \in L^2(\Omega)$, unique up to constants, such that $(u_\varepsilon, p_\varepsilon)$ is a weak solution of (2.2) and $u_\varepsilon = f$ on $\partial\Omega$. Moreover,*

$$\|u_\varepsilon\|_{H^1(\Omega)} + \|p_\varepsilon - \fint_\Omega p_\varepsilon\|_{L^2(\Omega)} \le C\Big\{\|F\|_{H^{-1}(\Omega)} + \|g\|_{L^2(\Omega)} + \|f\|_{H^{1/2}(\partial\Omega)}\Big\}, \tag{2.3}$$

where C depends only on d, μ, and Ω.

Theorem 2.3 is proved by using the Lax-Milgram Theorem. We mention that if Ω is $C^{1,1}$ and A is a constant matrix, the weak solution (u, p), given by Theorem 2.3, is in $H^2(\Omega; \mathbb{R}^d) \times H^1(\Omega)$, provided that $F \in L^2(\Omega; \mathbb{R}^d)$, $g \in H^1(\Omega)$ and $f \in H^{3/2}(\partial\Omega; \mathbb{R}^d)$. Moreover,

$$\|u\|_{H^2(\Omega)} + \|\nabla p\|_{L^2(\Omega)} \le C\Big\{\|F\|_{L^2(\Omega)} + \|g\|_{H^1(\Omega)} + \|f\|_{H^{3/2}(\partial\Omega)}\Big\}, \tag{2.4}$$

where C depends only on d, μ, and Ω (see e.g. [3]).

We denote by $H^1_{\text{per}}(Y; \mathbb{R}^d)$ the closure in $H^1(Y; \mathbb{R}^d)$ of $C^\infty_{\text{per}}(Y; \mathbb{R}^d)$, the set of C^∞ 1-periodic and $\mathbb{R}^d$-valued functions in $\mathbb{R}^d$. Let

$$a_{\text{per}}(\psi, \phi) = \int_Y a_{ij}^{\alpha\beta}(y) \frac{\partial \psi^\beta}{\partial x_j} \frac{\partial \phi^\alpha}{\partial x_i} dy,$$

where $\psi, \phi \in H^1_{\text{per}}(Y; \mathbb{R}^d)$. Define

$$V_{\text{per}}(Y) = \left\{ u \in H^1_{\text{per}}(Y; \mathbb{R}^d) : \ \operatorname{div}(u) = 0 \text{ in } Y \text{ and } \int_Y u = 0 \right\}.$$

By applying the Lax-Milgram Theorem to $a_{\text{per}}(\psi, \phi)$ on the Hilbert space $V_{\text{per}}(Y)$, one may show that for each $1 \le j, \beta \le d$, there exist 1-periodic functions $(\chi_j^\beta, \pi_j^\beta) \in H^1_{\text{loc}}(\mathbb{R}^d; \mathbb{R}^d) \times L^2_{\text{loc}}(\mathbb{R}^d)$, which are called the correctors for the Stokes system (2.2), such that

$$\begin{cases} \mathcal{L}_1(\chi_j^\beta + P_j^\beta) + \nabla \pi_j^\beta = 0 & \text{in } \mathbb{R}^d, \\ \operatorname{div} \chi_j^\beta = 0 & \text{in } \mathbb{R}^d, \\ \displaystyle\int_Y \pi_j^\beta = 0, \int_Y \chi_j^\beta = 0, \end{cases} \tag{2.5}$$

where $P_j^\beta = P_j^\beta(y) = y_j e^\beta = y_j(0, \cdots, 1, \cdots, 0)$ with 1 in the β^{th} position. Note that

$$\|\chi_j^\beta\|_{H^1(Y)} + \|\pi_j^\beta\|_{L^2(Y)} \le C,$$

where C depends only on d and μ. The homogenized system for the Stokes system (2.2) is given by

$$\begin{cases} \mathcal{L}_0(u_0) + \nabla p_0 = F \\ \operatorname{div} u_0 = g, \end{cases} \tag{2.6}$$

where $\mathcal{L}_0 = -\mathrm{div}(\widehat{A}\nabla)$ is a second-order elliptic operator with constant coefficients, and $\widehat{A} = (\widehat{a}_{ij}^{\alpha\beta})$, with

$$\widehat{a}_{ij}^{\alpha\beta} = a_{\mathrm{per}}(\chi_j^\beta + P_j^\beta, \chi_i^\alpha + P_i^\alpha).$$

We remark that $(\widehat{A})^* = \widehat{A^*}$, and the effective matrix $\widehat{A}$ satisfies the ellipticity condition $\mu|\xi|^2 \leq \widehat{a}_{ij}^{\alpha\beta}\xi_i^\alpha\xi_j^\beta \leq \mu_1|\xi|^2$, for any $\xi \in \mathbb{R}^{d\times d}$ and μ_1 depends only on d and μ. The following is a homogenization theorem for the Stokes system.

Theorem 2.4. *Suppose that $A(y)$ satisfies ellipticity condition (1.4) and periodicity condition (1.5). Let Ω be a bounded Lipschitz domain. Let $(u_\varepsilon, p_\varepsilon) \in H^1(\Omega;\mathbb{R}^d)\times L^2(\Omega)$ be a weak solution of (1.1), where $F \in H^{-1}(\Omega;\mathbb{R}^d)$, $g \in L^2(\Omega)$ and $f \in H^{1/2}(\partial\Omega;\mathbb{R}^d)$. Assume that $\int_\Omega p_\varepsilon = 0$. Then, as $\varepsilon \to 0$,*

$$\begin{cases} u_\varepsilon \to u_0 & \text{strongly in } L^2(\Omega;\mathbb{R}^d), \\ u_\varepsilon \rightharpoonup u_0 & \text{weakly in } H^1(\Omega;\mathbb{R}^d), \\ p_\varepsilon \rightharpoonup p_0 & \text{weakly in } L^2(\Omega), \\ A(x/\varepsilon)\nabla u_\varepsilon \rightharpoonup \widehat{A}\nabla u_0 & \text{weakly in } L^2(\Omega;\mathbb{R}^{d\times d}). \end{cases}$$

Moreover, (u_0, p_0) is the weak solution of the homogenized problem (1.6).

3 Convergence rates for u_ε in H^1

From now on we will assume that Ω is a bounded domain with boundary of class $C^{1,1}$, $F \in L^2(\Omega;\mathbb{R}^d)$, $g \in H^1(\Omega)$, and $f \in H^{3/2}(\partial\Omega;\mathbb{R}^d)$. We fix a linear continuous extension operator

$$E_\Omega : H^2(\Omega;\mathbb{R}^d) \to H^2(\mathbb{R}^d;\mathbb{R}^d),$$

and let

$$\widetilde{u}_0 = E_\Omega u_0, \tag{3.1}$$

so that $\widetilde{u}_0 = u_0$ in Ω and

$$\|\widetilde{u}_0\|_{H^2(\mathbb{R}^d)} \leq C\|u_0\|_{H^2(\Omega)}, \tag{3.2}$$

where C depends on Ω. We introduce a first order approximation of u_ε,

$$v_\varepsilon = u_0 + \varepsilon\chi^\varepsilon S_\varepsilon(\nabla\widetilde{u}_0).$$

Let $(w_\varepsilon, \tau_\varepsilon) \in H^1(\Omega;\mathbb{R}^d)\times L^2(\Omega)$ be a weak solution of

$$\begin{cases} \mathcal{L}_\varepsilon(w_\varepsilon) + \nabla\tau_\varepsilon = 0 & \text{in } \Omega, \\ \mathrm{div}(w_\varepsilon) = \varepsilon\ \mathrm{div}\big(\chi^\varepsilon S_\varepsilon\nabla\widetilde{u}_0\big) & \text{in } \Omega, \\ w_\varepsilon = \varepsilon\chi^\varepsilon S_\varepsilon(\nabla\widetilde{u}_0) & \text{on } \partial\Omega. \end{cases} \tag{3.3}$$

We will use w_ε to approximate the difference between u_ε and its first order approximation v_ε. To this end, for $1 \le i, j, \alpha, \beta \le d$, we let

$$b_{ij}^{\alpha\beta}(y) = a_{ij}^{\alpha\beta}(y) + a_{ik}^{\alpha\gamma}(y)\frac{\partial}{\partial y_k}\big(\chi_j^{\gamma\beta}\big) - \widehat{a}_{ij}^{\alpha\beta}. \tag{3.4}$$

Note that $b_{ij}^{\alpha\beta}$ is 1-periodic. By the definition of χ and $\widehat{A}$, $b_{ij}^{\alpha\beta} \in L^2(Y)$ satisfies

$$\int_Y b_{ij}^{\alpha\beta}(y)\, dy = 0.$$

and, for each $1 \le \alpha, \beta, j \le d$,

$$\begin{aligned}
\frac{\partial}{\partial y_i}\big(b_{ij}^{\alpha\beta}(y)\big) &= \frac{\partial}{\partial y_i}\big(a_{ij}^{\alpha\beta}(y)\big) + \frac{\partial}{\partial y_i}\left(a_{ik}^{\alpha\gamma}(y)\frac{\partial \chi_j^{\gamma\beta}}{\partial y_k}\right)\\
&= \frac{\partial}{\partial y_i}\big(a_{ij}^{\alpha\beta}(y)\big) - \frac{\partial}{\partial y_i}\left(a_{ik}^{\alpha\gamma}(y)\frac{\partial P_j^{\gamma\beta}}{\partial y_k}\right) + \frac{\partial}{\partial y_\alpha}(\pi_j^\beta)\\
&= \frac{\partial}{\partial y_\alpha}(\pi_j^\beta).
\end{aligned} \tag{3.5}$$

Lemma 3.1. *There exist $\Phi_{kij}^{\alpha\beta} \in H^1_{\text{per}}(Y)$ and $q_{ij}^\beta \in H^1_{\text{per}}(Y)$ such that*

$$b_{ij}^{\alpha\beta} = \frac{\partial}{\partial y_k}(\Phi_{kij}^{\alpha\beta}) + \frac{\partial}{\partial y_\alpha}(q_{ij}^\beta) \quad \text{and} \quad \Phi_{kij}^{\alpha\beta} = -\Phi_{ikj}^{\alpha\beta}. \tag{3.6}$$

Moreover,

$$\|\Phi_{kij}^{\alpha\beta}\|_{L^2(Y)} + \|q_{ij}^\beta\|_{L^2(Y)} \le C, \tag{3.7}$$

where C depends only on d and μ.

Proof. Fix $1 \le i, j, \beta \le d$. There exist $f_{ij}^\beta = (f_{ij}^{\alpha\beta}) \in H^2_{\text{per}}(Y; \mathbb{R}^d)$ and $q_{ij}^\beta \in H^1_{\text{per}}(Y)$ satisfying the following Stokes system,

$$\begin{cases}
\Delta f_{ij}^\beta + \nabla q_{ij}^\beta = b_{ij}^\beta & \text{in } Y,\\
\operatorname{div}(f_{ij}^\beta) = 0 & \text{in } Y,\\
\displaystyle\int_Y f_{ij}^\beta\, dy = 0,
\end{cases} \tag{3.8}$$

where $b_{ij}^\beta = (b_{ij}^{\alpha\beta})$. We now define

$$\Phi_{kij}^{\alpha\beta}(y) = \frac{\partial}{\partial y_k}(f_{ij}^{\alpha\beta}) - \frac{\partial}{\partial y_i}(f_{kj}^{\alpha\beta}).$$

Clearly, $\Phi_{kij}^{\alpha\beta} \in H^1_{\mathrm{per}}(Y)$ and $\Phi_{kij}^{\alpha\beta} = -\Phi_{ikj}^{\alpha\beta}$. Note that, by (3.5) and (3.8), $\frac{\partial f_{ij}^{\alpha\beta}}{\partial y_i} \in H^1_{\mathrm{per}}(Y)$ satisfies

$$\begin{cases} \Delta\left(\dfrac{\partial f_{ij}^{\alpha\beta}}{\partial y_i}\right) = -\dfrac{\partial}{\partial y_\alpha}\left(\dfrac{\partial q_{ij}^{\beta}}{\partial y_i}\right) + \dfrac{\partial b_{ij}^{\alpha\beta}}{\partial y_i} = \dfrac{\partial}{\partial y_\alpha}\left(\pi_j^\beta - \dfrac{\partial q_{ij}^{\beta}}{\partial y_i}\right), \\ \dfrac{\partial}{\partial y_\alpha}\left(\dfrac{\partial f_{ij}^{\alpha\beta}}{\partial y_i}\right) = 0. \end{cases} \tag{3.9}$$

It follows by the energy estimates that $\frac{\partial f_{ij}^{\alpha\beta}}{\partial y_i}$ is constant. Hence,

$$\frac{\partial}{\partial y_k}(\Phi_{kij}^{\alpha\beta}) = \frac{\partial^2}{\partial y_k \partial y_k}(f_{ij}^{\alpha\beta}) - \frac{\partial}{\partial y_i}\left(\frac{\partial}{\partial y_k}(f_{kj}^{\alpha\beta})\right) = b_{ij}^{\alpha\beta} - \frac{\partial}{\partial y_\alpha}(q_{ij}^\beta).$$

Furthermore, since $\|\chi_j^\beta\|_{H^1(Y)} \le C$, then

$$\|\Phi_{kij}^{\alpha\beta}\|_{L^2(Y)} + \|q_{ij}^\beta\|_{L^2(Y)} \le C\|b_{ij}^{\alpha\beta}\|_{L^2(Y)} \le C,$$

where C depends only on d and μ. This completes the proof. □

Remark 3.2. *Recall that π_j^β and q_{ij}^β are both 1-periodic. By (3.9) and the fact that $\frac{\partial f_{ij}^{\alpha\beta}}{\partial y_i}$ is constant, we see that π_j^β and $\frac{\partial q_{ij}^\beta}{\partial y_i}$ differ only by a constant. Since $\int_Y \pi_j^\beta = 0$, we obtain the following relation,*

$$\pi_j^\beta = \frac{\partial q_{ij}^\beta}{\partial y_i}. \tag{3.10}$$

Lemma 3.3. *Let Ω be a bounded $C^{1,1}$ domain. Suppose that A satisfies ellipticity condition (1.4) and periodicity condition (1.5). Given $g \in H^1(\Omega)$ and $f \in H^{3/2}(\partial\Omega;\mathbb{R}^d)$ satisfying the compatibility condition (1.2), for $F \in L^2(\Omega;\mathbb{R}^d)$, let $(u_\varepsilon, p_\varepsilon)$, (u_0, p_0) and $(w_\varepsilon, \tau_\varepsilon)$ be weak solutions of Dirichlet problems (1.1), (1.6) and (3.3), respectively. Then,*

$$\|u_\varepsilon - u_0 - \varepsilon\chi^\varepsilon S_\varepsilon(\nabla\widetilde{u}_0) + w_\varepsilon\|_{H_0^1(\Omega)} \le C\varepsilon\|u_0\|_{H^2(\Omega)}, \tag{3.11}$$

where C depends only on d, μ, and Ω.

Proof. Let

$$z_\varepsilon = u_\varepsilon - u_0 - \varepsilon\chi^\varepsilon S_\varepsilon(\nabla\widetilde{u}_0) + w_\varepsilon.$$

Then

$$\mathrm{div}(z_\varepsilon) = 0 \quad \text{in } \Omega \quad \text{and} \quad z_\varepsilon = 0 \quad \text{on } \partial\Omega.$$

Now we compute $\mathcal{L}_\varepsilon(z_\varepsilon)$,

$$
\begin{aligned}
(\mathcal{L}_\varepsilon(z_\varepsilon))^\alpha &= -\frac{\partial[p_\varepsilon - p_0 + \tau_\varepsilon]}{\partial x_\alpha} - \frac{\partial}{\partial x_i}\left(\left[\widehat{a}_{ij}^{\alpha\beta} - a_{ij}^{\alpha\beta}(x/\varepsilon)\right]\frac{\partial u_0^\beta}{\partial x_j}\right) \\
&\quad + \frac{\partial}{\partial x_i}\left(a_{ik}^{\alpha\gamma}(x/\varepsilon)\frac{\partial}{\partial x_k}\left[\varepsilon\chi_j^{\gamma\beta}(x/\varepsilon)\right]S_\varepsilon\frac{\partial\widetilde{u}_0^\beta}{\partial x_j}\right) + \varepsilon\frac{\partial}{\partial x_i}\left(a_{ik}^{\alpha\gamma}(x/\varepsilon)\chi_j^{\gamma\beta}(x/\varepsilon)S_\varepsilon\frac{\partial^2\widetilde{u}_0^\beta}{\partial x_k\partial x_j}\right) \\
&= -\frac{\partial[p_\varepsilon - p_0 + \tau_\varepsilon]}{\partial x_\alpha} - \frac{\partial}{\partial x_i}\left(\left[\widehat{a}_{ij}^{\alpha\beta} - a_{ij}^{\alpha\beta}(x/\varepsilon)\right]\left[\frac{\partial u_0^\beta}{\partial x_j} - S_\varepsilon\frac{\partial\widetilde{u}_0^\beta}{\partial x_j}\right]\right) \\
&\quad + \frac{\partial}{\partial x_i}\left(b_{ij}^{\alpha\beta}(x/\varepsilon)S_\varepsilon\frac{\partial\widetilde{u}_0^\beta}{\partial x_j}\right) + \varepsilon\frac{\partial}{\partial x_i}\left(a_{ik}^{\alpha\gamma}(x/\varepsilon)\chi_j^{\gamma\beta}(x/\varepsilon)S_\varepsilon\frac{\partial^2\widetilde{u}_0^\beta}{\partial x_k\partial x_j}\right).
\end{aligned}
$$

Using Lemma 3.1, we may write

$$
\begin{aligned}
\frac{\partial}{\partial x_i}\left(b_{ij}^{\alpha\beta}(x/\varepsilon)S_\varepsilon\frac{\partial\widetilde{u}_0^\beta}{\partial x_j}\right) &= \frac{\partial}{\partial x_i}\left(\left[\frac{\partial}{\partial x_k}\left(\varepsilon\Phi_{kij}^{\alpha\beta}(x/\varepsilon)\right) + \frac{\partial}{\partial x_\alpha}\left(\varepsilon q_{ij}^\beta(x/\varepsilon)\right)\right]S_\varepsilon\frac{\partial\widetilde{u}_0^\beta}{\partial x_j}\right) \\
&= I_1 + I_2.
\end{aligned} \tag{3.12}
$$

Since $\Phi_{kij}^{\alpha\beta} = -\Phi_{ikj}^{\alpha\beta}$, we see that

$$
\begin{aligned}
I_1 &= \frac{\partial^2}{\partial x_i\partial x_k}\left(\varepsilon\Phi_{kij}^{\alpha\beta}(x/\varepsilon)S_\varepsilon\frac{\partial\widetilde{u_0}^\beta}{\partial x_j}\right) - \varepsilon\frac{\partial}{\partial x_i}\left(\Phi_{kij}^{\alpha\beta}(x/\varepsilon)S_\varepsilon\frac{\partial^2\widetilde{u}_0^\beta}{\partial x_j\partial x_k}\right) \\
&= -\varepsilon\frac{\partial}{\partial x_i}\left(\Phi_{kij}^{\alpha\beta}(x/\varepsilon)S_\varepsilon\frac{\partial^2\widetilde{u}_0^\beta}{\partial x_j\partial x_k}\right).
\end{aligned}
$$

For the second term in the RHS of (3.12), we have

$$
\begin{aligned}
I_2 &= \frac{\partial}{\partial x_\alpha}\left(\frac{\partial}{\partial x_i}\left[\varepsilon q_{ij}^\beta(x/\varepsilon)S_\varepsilon\frac{\partial\widetilde{u}_0^\beta}{\partial x_j}\right]\right) - \frac{\partial}{\partial x_i}\left(\varepsilon q_{ij}^\beta(x/\varepsilon)S_\varepsilon\frac{\partial^2\widetilde{u}_0^\beta}{\partial x_\alpha\partial x_j}\right) \\
&= I_3 - \frac{\partial}{\partial x_i}\left(\varepsilon q_{ij}^\beta(x/\varepsilon)S_\varepsilon\frac{\partial^2\widetilde{u}_0^\beta}{\partial x_\alpha\partial x_j}\right).
\end{aligned} \tag{3.13}
$$

In view of (3.10), for the first term on the RHS of (3.13), we obtain

$$
I_3 = \frac{\partial}{\partial x_\alpha}\left(\pi_j^\beta(x/\varepsilon)S_\varepsilon\frac{\partial\widetilde{u}_0^\beta}{\partial x_j}\right) + \frac{\partial}{\partial x_\alpha}\left(\varepsilon q_{ij}^\beta(x/\varepsilon)S_\varepsilon\frac{\partial^2\widetilde{u}_0^\beta}{\partial x_j\partial x_i}\right). \tag{3.14}
$$

Putting altogether, we have shown that

$$
\begin{aligned}
&(\mathcal{L}_\varepsilon(z_\varepsilon))^\alpha + \frac{\partial}{\partial x_\alpha}\left(p_\varepsilon - p_0 - \pi_j^\beta(x/\varepsilon) S_\varepsilon \frac{\partial \widetilde{u}_0^\beta}{\partial x_j} - \varepsilon q_{ij}^\beta(x/\varepsilon) S_\varepsilon \frac{\partial^2 \widetilde{u}_0^\beta}{\partial x_j \partial x_i} + \tau_\varepsilon \right) \\
&= \varepsilon \frac{\partial}{\partial x_i}\left(\left[a_{ij}^{\alpha\gamma}(x/\varepsilon)\chi_k^{\gamma\beta}(x/\varepsilon) - \Phi_{kij}^{\alpha\beta}(x/\varepsilon)\right] S_\varepsilon \frac{\partial^2 \widetilde{u}_0^\beta}{\partial x_j \partial x_k}\right) \\
&\qquad - \varepsilon \frac{\partial}{\partial x_i}\left(q_{ij}^\beta(x/\varepsilon) S_\varepsilon \frac{\partial^2 \widetilde{u}_0^\beta}{\partial x_\alpha \partial x_j}\right) \\
&\qquad - \frac{\partial}{\partial x_i}\left(\left[\widehat{a}_{ij}^{\alpha\beta} - a_{ij}^{\alpha\beta}(x/\varepsilon)\right]\left[\frac{\partial u_0^\beta}{\partial x_j} - S_\varepsilon \frac{\partial \widetilde{u}_0^\beta}{\partial x_j}\right]\right).
\end{aligned}
\tag{3.15}
$$

Since $z_\varepsilon \in H_0^1(\Omega;\mathbb{R}^d)$ and $\operatorname{div}(z_\varepsilon) = 0$ in Ω, it follows from (3.15) by the energy estimate (2.3) that

$$
\begin{aligned}
c\int_\Omega |\nabla z_\varepsilon|^2 dx \le \varepsilon^2 \int_\Omega \Big| \big[|\chi(x/\varepsilon)| + |\Phi(x/\varepsilon)| \big] S_\varepsilon(\nabla^2 \widetilde{u}_0)\Big|^2 dx \\
+ \varepsilon^2 \int_\Omega \Big| q(x/\varepsilon) S_\varepsilon(\nabla^2 \widetilde{u}_0)\Big|^2 dx + \int_\Omega \Big| \nabla u_0 - S_\varepsilon(\nabla \widetilde{u}_0)\Big|^2 dx.
\end{aligned}
$$

Now we apply Propositions 2.1-2.2 as well as (3.2). This gives

$$
\begin{aligned}
\|\nabla z_\varepsilon\|_{L^2(\Omega)} &\le C\varepsilon \left(\|\chi\|_{L^2(Y)} + \|\Phi\|_{L^2(Y)} + \|q\|_{L^2(Y)} + 1\right) \|\nabla^2 \widetilde{u}_0\|_{L^2(\mathbb{R}^d)} \\
&\le C\varepsilon \|\nabla^2 \widetilde{u}_0\|_{L^2(\mathbb{R}^d)} \\
&\le C\varepsilon \|u_0\|_{H^2(\Omega)},
\end{aligned}
$$

where C depends only on d, μ and Ω. Hence we have proved the desired result, $\|z_\varepsilon\|_{H_0^1(\Omega)} \le C\varepsilon\|u_0\|_{H^2(\Omega)}$, and completed the proof. $\square$

For $r > 0$, let

$$
\begin{aligned}
(\partial\Omega)_r &= \{x \in \mathbb{R}^d : \operatorname{dist}(x, \partial\Omega) \le r\}, \\
\Omega_r &= \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) \le r\}.
\end{aligned}
$$

We choose two cut-off functions $\theta_\varepsilon(x)$ and $\widetilde{\theta}_\varepsilon(x)$ in $\mathbb{R}^d$ satisfying the following conditions,

$$
\begin{aligned}
&\theta_\varepsilon \in C_0^\infty(\mathbb{R}^d), \quad \operatorname{supp}(\theta_\varepsilon) \subset (\partial\Omega)_\varepsilon, \quad 0 \le \theta_\varepsilon(x) \le 1, \\
&\theta_\varepsilon|_{\partial\Omega} = 1, \quad |\nabla\theta_\varepsilon| \le \kappa/\varepsilon,
\end{aligned}
\tag{3.16}
$$

and

$$
\begin{aligned}
&\widetilde{\theta}_\varepsilon \in C_0^\infty(\mathbb{R}^d), \quad \operatorname{supp}(\widetilde{\theta}_\varepsilon) \subset (\partial\Omega)_{2\varepsilon}, \quad 0 \le \widetilde{\theta}_\varepsilon(x) \le 1, \\
&\widetilde{\theta}_\varepsilon(x) = 1 \text{ for } x \in (\partial\Omega)_\varepsilon, \quad |\nabla\widetilde{\theta}_\varepsilon| \le \widetilde{\kappa}/\varepsilon.
\end{aligned}
\tag{3.17}
$$

The following is an estimate for integrals near the boundary, see [14] for example.

Lemma 3.4. *Let $\Omega \subset \mathbb{R}^d$ be a bounded C^1 domain. Then, for any function $u \in H^1(\Omega)$,*

$$\int_{\Omega_r} |u|^2 dx \le Cr\|u\|_{H^1(\Omega)}\|u\|_{L^2(\Omega)}.$$

Moreover, for any 1-periodic function $f \in L^2(Y)$ and $u \in H^1(\mathbb{R}^d)$,

$$\int_{(\partial\Omega)_{2\varepsilon}} |f^\varepsilon|^2 |S_\varepsilon u|^2 dx \le C\varepsilon \|f\|_{L^2(Y)} \|u\|_{H^1(\mathbb{R}^d)} \|u\|_{L^2(\mathbb{R}^d)},$$

where C depends only on Ω.

We are now ready to give the proof of (1.8).

Proof of estimate (1.8). By Lemma 3.3, the problem has been reduced to estimating w_ε in H^1. Notice that by the energy estimate (2.3),

$$\begin{aligned}
&\|w_\varepsilon\|_{H^1(\Omega)} \le C\varepsilon\|\chi^\varepsilon S_\varepsilon \nabla\widetilde{u}_0\|_{H^{1/2}(\partial\Omega)} + C\varepsilon\|\mathrm{div}\big(\chi^\varepsilon S_\varepsilon \nabla\widetilde{u}_0)\|_{L^2(\Omega)}\\
&\le C\varepsilon\|\theta_\varepsilon\chi^\varepsilon S_\varepsilon\nabla\widetilde{u}_0\|_{H^1(\Omega)} + C\varepsilon\|\chi^\varepsilon\nabla S_\varepsilon(\nabla\widetilde{u}_0)\|_{L^2(\Omega)}\\
&\le C\varepsilon\Big\{\|\chi^\varepsilon S_\varepsilon(\nabla\widetilde{u}_0)\|_{L^2(\Omega)} + \|(\nabla\theta_\varepsilon)\chi^\varepsilon S_\varepsilon(\nabla\widetilde{u}_0)\|_{L^2(\Omega)}\\
&\qquad\qquad + \varepsilon^{-1}\|\theta_\varepsilon(\nabla\chi)^\varepsilon S_\varepsilon(\nabla\widetilde{u}_0)\|_{L^2(\Omega)} + \|\chi^\varepsilon S_\varepsilon(\nabla^2\widetilde{u}_0)\|_{L^2(\Omega)}\Big\}\\
&\le C\varepsilon\Big\{\|\widetilde{u}_0\|_{H^2(\mathbb{R}^d)} + \varepsilon^{-1}\|\chi^\varepsilon S_\varepsilon(\nabla\widetilde{u}_0)\|_{L^2(\Omega_\varepsilon)} + \varepsilon^{-1}\|(\nabla\chi)^\varepsilon S_\varepsilon(\nabla\widetilde{u}_0)\|_{L^2(\Omega_\varepsilon)}\Big\}\\
&\le C\varepsilon^{1/2}\|\widetilde{u}_0\|_{H^2(\mathbb{R}^d)},
\end{aligned} \tag{3.18}$$

where we have used Proposition 2.2 for the fourth inequality and Lemma 3.4 for the last. We point out that the fact $\mathrm{div}(\chi) = 0$ in $\mathbb{R}^d$ is also used for the second inequality in (3.18). Therefore,

$$\begin{aligned}
\|u_\varepsilon - u_0 - \varepsilon\chi^\varepsilon S_\varepsilon\left(\nabla\widetilde{u}_0\right)\|_{H^1(\Omega)} &\le \|z_\varepsilon\|_{H^1(\Omega)} + \|w_\varepsilon\|_{H^1(\Omega)}\\
&\le C\sqrt{\varepsilon}\|u_0\|_{H^2(\Omega)},
\end{aligned}$$

where C depends only on d, μ, and Ω. This completes the proof. □

4 Convergence rates for the pressure term

To prove estimate(1.9), we first recall that if $(u_\varepsilon, p_\varepsilon) \in H^1(\Omega;\mathbb{R}^d) \times L^2(\Omega)$ is a weak solution of the Stokes system (1.1), then

$$\|p_\varepsilon - \fint_\Omega p_\varepsilon\|_{L^2(\Omega)} \le C\|\nabla p_\varepsilon\|_{H^{-1}(\Omega)} \le C\Big\{\|F\|_{H^{-1}(\Omega)} + \|u_\varepsilon\|_{H^1(\Omega)}\Big\}, \tag{4.1}$$

where C depends only on d, μ, and Ω (see e.g. [15]).

Proof of estimate (1.9). Since $\int_\Omega p_\varepsilon = \int_\Omega p_0 = 0$, using (4.1) and (3.15), we see that

$$\begin{aligned}
&\|p_\varepsilon - p_0 - \Big[\big(\pi^\varepsilon S_\varepsilon \nabla \widetilde{u}_0 + \varepsilon q^\varepsilon S_\varepsilon \nabla^2 \widetilde{u}_0 - \tau_\varepsilon\big) - \fint_\Omega \big(\pi^\varepsilon S_\varepsilon \nabla \widetilde{u}_0 + \varepsilon q^\varepsilon S_\varepsilon \nabla^2 \widetilde{u}_0 - \tau_\varepsilon\big)\Big]\|_{L^2(\Omega)} \\
&\le C\|\nabla \Big[p_\varepsilon - p_0 - \pi^\varepsilon S_\varepsilon\left(\nabla \widetilde{u}_0\right) - \varepsilon q^\varepsilon S_\varepsilon\left(\nabla^2 \widetilde{u}_0\right) + \tau_\varepsilon\Big]\|_{H^{-1}(\Omega)} \\
&\le C\left\{\|\nabla z_\varepsilon\|_{L^2(\Omega)} + \varepsilon\left\|\big(|\chi^\varepsilon| + |\Phi^\varepsilon| + |q^\varepsilon|\big) S_\varepsilon\left(\nabla^2 \widetilde{u}_0\right)\right\|_{L^2(\Omega)} + \|S_\varepsilon\left(\nabla \widetilde{u}_0\right) - \nabla u_0\|_{L^2(\Omega)}\right\} \\
&\le C\varepsilon \|u_0\|_{H^2(\Omega)},
\end{aligned} \tag{4.2}$$

where the last inequality follows from the proof of Lemma 3.3. Note that by Propostion 2.2 and (3.2),

$$\varepsilon \|q^\varepsilon S_\varepsilon \nabla^2 \widetilde{u}_0 - \fint_\Omega q^\varepsilon S_\varepsilon \nabla^2 \tilde{u}_0\|_{L^2(\Omega)} \le C\varepsilon \|\widetilde{u}_0\|_{H^2(\mathbb{R}^d)} \le C\varepsilon \|u_0\|_{H^2(\Omega)}. \tag{4.3}$$

Also, by the definition of $(w_\varepsilon, \tau_\varepsilon)$ and (4.1),

$$\|\tau_\varepsilon - \fint_\Omega \tau_\varepsilon\|_{L^2(\Omega)} \le C\|\nabla \tau_\varepsilon\|_{H^{-1}(\Omega)} \le C\|\nabla w_\varepsilon\|_{L^2(\Omega)} \le C\sqrt{\varepsilon}\|u_0\|_{H^2(\Omega)}, \tag{4.4}$$

where the last inequality follows from (3.18). By combining (4.2), (4.3) and (4.4), we have proved that

$$\|p_\varepsilon - p_0 - \Big[\pi^\varepsilon S_\varepsilon\left(\nabla \widetilde{u}_0\right) - \fint_\Omega \pi^\varepsilon S_\varepsilon\left(\nabla \widetilde{u}_0\right)\Big]\|_{L^2(\Omega)} \le C\sqrt{\varepsilon}\|u_0\|_{H^2(\Omega)}.$$

This completes the proof. $\square$

5 Convergence rates for u_ε in L^2

To establish the sharp $O(\varepsilon)$ rate for u_ε in L^2, in view of (3.11), we obtain

$$\|u_\varepsilon - u_0 - \varepsilon \chi^\varepsilon S_\varepsilon \nabla \widetilde{u}_0 + w_\varepsilon\|_{L^2(\Omega)} \le C\varepsilon \|u_0\|_{H^2(\Omega)}.$$

Using Proposition 2.2 and (3.2),

$$\|\chi^\varepsilon S_\varepsilon \nabla \widetilde{u}_0\|_{L^2(\Omega)} \le C\|\chi\|_{L^2(Y)}\|\nabla \widetilde{u}_0\|_{L^2(\mathbb{R}^d)} \le C\|u_0\|_{H^2(\Omega)}.$$

Thus,

$$\|u_\varepsilon - u_0\|_{L^2(\Omega)} \le C\varepsilon \|u_0\|_{H^2(\Omega)} + \|w_\varepsilon\|_{L^2(\Omega)}, \tag{5.1}$$

and it remains to estimate $\|w_\varepsilon\|_{L^2(\Omega)}$.

Lemma 5.1. *Let Ω be a bounded $C^{1,1}$ domain. Suppose that A satisfies ellipticity condition (1.4) and periodicity condition (1.5). Given $g \in H^1(\Omega)$ and $f \in H^{3/2}(\partial\Omega; \mathbb{R}^d)$*

satisfying the compatibility condition (1.2), for $F \in L^2(\Omega;\mathbb{R}^d)$*, let* $(u_\varepsilon, p_\varepsilon)$*,* (u_0, p_0) *be weak solutions of the Dirichlet problems (1.1), (1.6), respectively. Then*

$$\|u_\varepsilon - u_0 - \varepsilon(1-\widetilde{\theta}_\varepsilon)\chi^\varepsilon S_\varepsilon\left(\nabla \widetilde{u}_0\right)\|_{H^1(\Omega)} \leq C\sqrt{\varepsilon}\|u_0\|_{H^2(\Omega)}, \tag{5.2}$$

and

$$\|p_\varepsilon - p_0 - \Big[(1-\widetilde{\theta}_\varepsilon)\pi^\varepsilon S_\varepsilon\left(\nabla \widetilde{u}_0\right) - \fint_\Omega \pi^\varepsilon S_\varepsilon\left(\nabla \widetilde{u}_0\right)\Big]\|_{L^2(\Omega)} \leq C\sqrt{\varepsilon}\|u_0\|_{H^2(\Omega)}, \tag{5.3}$$

where C *depends only on* d*,* μ*, and* Ω*.*

Proof. Note that

$$\begin{aligned}\|\varepsilon\widetilde{\theta}_\varepsilon\chi^\varepsilon S_\varepsilon\nabla\widetilde{u}_0\|_{H^1(\Omega)} &\leq C\varepsilon\|\chi^\varepsilon S_\varepsilon(\nabla\widetilde{u}_0)\|_{L^2(\Omega_{2\varepsilon})} + C\varepsilon\|\chi^\varepsilon S_\varepsilon(\nabla^2\widetilde{u}_0)\|_{L^2(\Omega_{2\varepsilon})}\\ &\qquad + C\|(|\chi^\varepsilon| + |(\nabla\chi)^\varepsilon|)S_\varepsilon(\nabla\widetilde{u}_0)\|_{L^2(\Omega_{2\varepsilon})}\\ &\leq C\sqrt{\varepsilon}\|u_0\|_{H^2(\Omega)},\end{aligned} \tag{5.4}$$

where we have used Lemma 3.4 and Proposition (2.2) for the last inequality. This, together with estimate (1.8), gives (5.2).

Similarly, using Lemma 3.4, we see that

$$\|\widetilde{\theta}_\varepsilon\pi^\varepsilon S_\varepsilon\nabla\widetilde{u}_0\|^2_{L^2(\Omega)} \leq C\int_{(\partial\Omega)_{2\varepsilon}} |\pi^\varepsilon S_\varepsilon(\nabla\widetilde{u}_0)|^2 \leq C\varepsilon\|u_0\|^2_{H^2(\Omega)}.$$

This, together with estimate (1.9), gives (5.3). □

Proof of Theorem 1.1. In view of (5.1), it suffices to show that

$$\|w_\varepsilon\|_{L^2(\Omega)} \leq C\varepsilon\|u_0\|_{H^2(\Omega)}.$$

Furthermore, let

$$\phi_\varepsilon = \varepsilon\theta_\varepsilon\chi^\varepsilon S_\varepsilon\nabla\widetilde{u}_0.$$

Since $\|\phi_\varepsilon\|_{L^2(\Omega)} \leq C\varepsilon\|u_0\|_{H^2(\Omega)}$, it is enough to show that

$$\|\eta_\varepsilon\|_{L^2(\Omega)} \leq C\varepsilon\|u_0\|_{H^2(\Omega)}, \tag{5.5}$$

where $\eta_\varepsilon = w_\varepsilon - \phi_\varepsilon$.

To this end, we first note that by the definition of $(w_\varepsilon, \tau_\varepsilon)$ in (3.3), the functions $(\eta_\varepsilon, \tau_\varepsilon) \in H^1_0(\Omega;\mathbb{R}^d) \times L^2(\Omega)$ satisfy

$$\begin{cases} \mathcal{L}_\varepsilon(\eta_\varepsilon) + \nabla\tau_\varepsilon = -\mathcal{L}_\varepsilon\phi_\varepsilon & \text{in } \Omega,\\ \qquad \text{div } \eta_\varepsilon = \varepsilon\,\text{div}\left((1-\theta_\varepsilon)\chi^\varepsilon S_\varepsilon\nabla\widetilde{u}_0\right) & \text{in } \Omega,\\ \qquad\quad \eta_\varepsilon = 0 & \text{on } \partial\Omega.\end{cases} \tag{5.6}$$

Let $(\eta_0, \tau_0) \in H_0^1(\Omega; \mathbb{R}^d) \times L^2(\Omega)$ be a weak solution of the homogenized Dirichlet problem

$$\begin{cases} \mathcal{L}_0(\eta_0) + \nabla \tau_0 = -\mathcal{L}_\varepsilon \phi_\varepsilon & \text{in } \Omega, \\ \text{div } \eta_0 = \varepsilon \,\text{div } ((1-\theta_\varepsilon)\chi^\varepsilon S_\varepsilon \nabla \widetilde{u}_0) & \text{in } \Omega, \\ \eta_0 = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.7}$$

To estimate $\eta_\varepsilon - \eta_0$, we consider the following duality problems. For any $H \in L^2(\Omega; \mathbb{R}^d)$, let $(\rho_\varepsilon, \sigma_\varepsilon) \in H_0^1(\Omega; \mathbb{R}^d) \times L^2(\Omega)$ be the weak solution of

$$\begin{cases} \mathcal{L}_\varepsilon^*(\rho_\varepsilon) + \nabla \sigma_\varepsilon = H & \text{in } \Omega, \\ \text{div } \rho_\varepsilon = 0 & \text{in } \Omega, \\ \rho_\varepsilon = 0 & \text{on } \partial\Omega, \end{cases} \tag{5.8}$$

and $(\rho_0, \sigma_0) \in (H^2(\Omega; \mathbb{R}^d) \cap H_0^1(\Omega; \mathbb{R}^d)) \times H^1(\Omega)$ the weak solution of

$$\begin{cases} \mathcal{L}_0^*(\rho_0) + \nabla \sigma_0 = H & \text{in } \Omega, \\ \text{div } \rho_0 = 0 & \text{in } \Omega, \\ \rho_0 = 0 & \text{on } \partial\Omega, \end{cases} \tag{5.9}$$

with

$$\int_\Omega \sigma_\varepsilon = \int_\Omega \sigma_0 = 0.$$

Here we have used the notation: $\mathcal{L}_\varepsilon^* = -\text{div}\big(A^*(x/\varepsilon)\nabla\big)$ and $\mathcal{L}_0^* = -\text{div}\big(\widehat{A^*}\nabla\big)$. We note that Lemma 5.1 continues to hold for $\mathcal{L}_\varepsilon^*$, as A^* satisfies the same conditions as A. Also, by the $W^{2,2}$ estimates (2.4) for Stokes systems with constant coefficients in $C^{1,1}$ domains,

$$\|\rho_0\|_{H^2(\Omega)} + \|\sigma_0\|_{H^1(\Omega)} \le C\, \|H\|_{L^2(\Omega)}.$$

As a result, we have

$$\|\rho_\varepsilon - \rho_0 - \varepsilon(1-\widetilde{\theta}_\varepsilon)\chi^{*\varepsilon} S_\varepsilon\,(\nabla\widetilde{\rho}_0)\,\|_{H^1(\Omega)} \le C\sqrt{\varepsilon}\|\rho_0\|_{H^2(\Omega)} \le C\sqrt{\varepsilon}\|H\|_{L^2(\Omega)}, \tag{5.10}$$

and

$$\|\sigma_\varepsilon - \sigma_0 - \Big[(1-\widetilde{\theta}_\varepsilon)\pi^{*\varepsilon} S_\varepsilon\,(\nabla\widetilde{\rho}_0) - \fint_\Omega \pi^{*\varepsilon} S_\varepsilon\,(\nabla\widetilde{\rho}_0)\Big]\|_{L^2(\Omega)} \le C\sqrt{\varepsilon}\|H\|_{L^2(\Omega)}, \tag{5.11}$$

where (χ^*, π^*) denotes the correctors associated with the adjoint matrix A^*.

Let $\Psi = -\mathcal{L}_\varepsilon \phi_\varepsilon$, and

$$\Gamma = \text{div } (\varepsilon(1-\theta_\varepsilon)\chi^\varepsilon S_\varepsilon \nabla\widetilde{u}_0).$$

Note that by (5.6), (5.7), (5.8) and (5.9),

$$\begin{aligned} \int_\Omega H \cdot (\eta_\varepsilon - \eta_0) &= \langle \Psi, \rho_\varepsilon - \rho_0 \rangle_{H^{-1}(\Omega;\mathbb{R}^d)\times H_0^1(\Omega;\mathbb{R}^d)} - \int_\Omega \Gamma(\sigma_\varepsilon - \sigma_0) \\ &= J_1 + J_2. \end{aligned} \tag{5.12}$$

For the first term of the RHS of (5.12), because $\Psi \in H^{-1}(\Omega;\mathbb{R}^d)$ is supported in $(\partial\Omega)_\varepsilon$, and $1-\widetilde{\theta}_\varepsilon = 0$ in $(\partial\Omega)_\varepsilon$, we obtain

$$J_1 = \langle \Psi, \rho_\varepsilon - \rho_0 - \varepsilon(1-\widetilde{\theta}_\varepsilon)\chi^{*\varepsilon} S_\varepsilon\left(\nabla\widetilde{\rho}_0\right)\rangle_{H^{-1}(\Omega;\mathbb{R}^d)\times H_0^1(\Omega;\mathbb{R}^d)}.$$

Therefore,

$$\begin{aligned}
|J_1| &\le \|\Psi\|_{H^{-1}(\Omega)} \|\rho_\varepsilon - \rho_0 - \varepsilon(1-\widetilde{\theta}_\varepsilon)\chi^{*\varepsilon} S_\varepsilon\left(\nabla\widetilde{\rho}_0\right)\|_{H^1(\Omega)} \\
&\le C\|\varepsilon\theta_\varepsilon \chi^\varepsilon S_\varepsilon \nabla\widetilde{u}_0\|_{H^1(\Omega)}\sqrt{\varepsilon}\|H\|_{L^2(\Omega)} \\
&\le C\varepsilon\|u_0\|_{H^2(\Omega)}\|H\|_{L^2(\Omega)}
\end{aligned} \tag{5.13}$$

where the second inequality follows from (5.10), and the last inequality follows from the analog of (5.4) (with $\widetilde{\theta}_\varepsilon$ replaced by θ_ε). For the second term of the RHS of (5.12), we recall that div $(\chi) = 0$. Hence,

$$\Gamma = -\varepsilon\frac{\partial\theta_\varepsilon}{\partial x_\alpha}\chi_j^{\alpha\beta}(x/\varepsilon)S_\varepsilon\frac{\partial\widetilde{u}_0^\beta}{\partial x_j} + \varepsilon(1-\theta_\varepsilon)\chi_j^{\alpha\beta}(x/\varepsilon)S_\varepsilon\frac{\partial^2\widetilde{u}_0^\beta}{\partial x_\alpha\partial x_j} = \Gamma_1+\Gamma_2.$$

Since $\int_\Omega \Gamma = 0$, for any constant E,

$$J_2 = -\int_\Omega \Gamma(\sigma_\varepsilon - \sigma_0 + E) = -\int_\Omega [\Gamma_1+\Gamma_2](\sigma_\varepsilon-\sigma_0+E).$$

We split J_2 as two integrals, for the first integral, again since $1-\widetilde{\theta}_\varepsilon = 0$ in $(\partial\Omega)_\varepsilon$ and Γ_1 is supported in $(\partial\Omega)_\varepsilon$, just as we did for J_1,

$$-\int_\Omega \Gamma_1(\sigma_\varepsilon-\sigma_0+E) = -\int_\Omega \Gamma_1\Big(\sigma_\varepsilon - \sigma_0 - (1-\widetilde{\theta}_\varepsilon)\pi^{*\varepsilon}S_\varepsilon\left(\nabla\tilde{\rho}_0\right) + E\Big).$$

Now, if we choose the constant E as $E = \fint_\Omega \pi^{*\varepsilon}S_\varepsilon\left(\nabla\tilde{\rho}_0\right)$, then

$$\begin{aligned}
&\left|\int_\Omega \Gamma_1(\sigma_\varepsilon-\sigma_0+E)\right| \\
&= \left|\int_\Omega \Gamma_1\Big\{\sigma_\varepsilon-\sigma_0 - \Big[(1-\widetilde{\theta}_\varepsilon)\pi^{*\varepsilon}S_\varepsilon\left(\nabla\widetilde{\rho}_0\right) - \fint_\Omega \pi^{*\varepsilon}S_\varepsilon\left(\nabla\widetilde{\rho}_0\right)\Big]\Big\}\right| \\
&\le C\|\Gamma_1\|_{L^2((\partial\Omega)_\varepsilon)}\sqrt{\varepsilon}\|H\|_{L^2(\Omega)} \\
&\le C\big(\sqrt{\varepsilon}\|\chi\|_{L^2(Y)}\|\nabla\widetilde{u}_0\|_{H^1(\mathbb{R}^d)}\big)\big(\sqrt{\varepsilon}\|H\|_{L^2(\Omega)}\big) \\
&\le C\varepsilon\|u_0\|_{H^2(\Omega)}\|H\|_{L^2(\Omega)},
\end{aligned} \tag{5.14}$$

where we have used (5.11) and Lemma 3.4. For the second integral in J_2, we have

$$\begin{aligned}
&\left|\int_\Omega \Gamma_2\big(\sigma_\varepsilon-\sigma_0+E\big)\right| \\
&\le \|\Gamma_2\|_{L^2(\Omega)}\|\sigma_\varepsilon-\sigma_0+\fint_\Omega \pi^{*\varepsilon}S_\varepsilon\left(\nabla\widetilde{\rho}_0\right)\|_{L^2(\Omega)} \\
&\le C\varepsilon\|u_0\|_{H^2(\Omega)}\|H\|_{L^2(\Omega)},
\end{aligned} \tag{5.15}$$

where for the last inequality we have used

$$\begin{aligned}\|\sigma_\varepsilon - \sigma_0 + \fint_\Omega \pi^{*\varepsilon} S_\varepsilon\left(\nabla\widetilde{\rho}_0\right)\|_{L^2(\Omega)} &\le \|\sigma_\varepsilon\|_{L^2(\Omega)} + \|\sigma_0\|_{L^2(\Omega)} + \|\fint_\Omega \pi^{*\varepsilon} S_\varepsilon\left(\nabla\widetilde{\rho}_0\right)\|_{L^2(\Omega)}\\ &\le C\|H\|_{L^2(\Omega)}.\end{aligned}$$

Therefore, by combining (5.13)-(5.15), we have proved

$$\left|\int_\Omega H(\eta_\varepsilon - \eta_0)\right| \le C\varepsilon\|u_0\|_{H^2(\Omega)}\|H\|_{L^2(\Omega)} \quad \text{for any } H \in L^2(\Omega;\mathbb{R}^d). \tag{5.16}$$

By duality this implies that

$$\|\eta_\varepsilon - \eta_0\|_{L^2(\Omega)} \le C\varepsilon\|u_0\|_{H^2(\Omega)}. \tag{5.17}$$

Finally, the problem has been reduced to the estimate of $\|\eta_0\|_{L^2(\Omega)}$. This will be done by another duality argument. Let (ρ_0, σ_0) be defined by (5.9). Then

$$\begin{aligned}&\left|\int_\Omega H\cdot\eta_0\right| = \left|\langle\Psi,\rho_0\rangle_{H^{-1}(\Omega;\mathbb{R}^d)\times H_0^1(\Omega;\mathbb{R}^d)} - \int_\Omega \Gamma\sigma_0\right|\\ &\le \left|\langle\Psi,\rho_0\rangle_{H^{-1}(\Omega;\mathbb{R}^d)\times H_0^1(\Omega;\mathbb{R}^d)}\right| + \left|\int_{\Omega_\varepsilon}\Gamma_1\sigma_0\right| + \left|\int_\Omega \Gamma_2\sigma_0\right|\\ &= K_1 + K_2 + K_3,\end{aligned} \tag{5.18}$$

where Ψ, Γ, Γ_1 and Γ_2 are as denoted above. Notice that again by Lemma 3.4 and the analog of (5.4) (with $\widetilde{\theta}_\varepsilon$ replaced by θ_ε), we have

$$\begin{aligned}K_1 &\le \|\Psi\|_{H^{-1}(\Omega)}\|\rho_0\|_{H^1(\Omega_\varepsilon)}\\ &\le C\|\varepsilon\theta_\varepsilon\chi^\varepsilon S_\varepsilon\nabla\widetilde{u}_0\|_{H^1(\Omega)}\sqrt{\varepsilon}\|\rho_0\|_{H^2(\Omega)}\\ &\le C(\sqrt{\varepsilon}\|u_0\|_{H^2(\Omega)})(\sqrt{\varepsilon}\|\rho_0\|_{H^2(\Omega)})\\ &\le C\varepsilon\|u_0\|_{H^2(\Omega)}\|H\|_{L^2(\Omega)}.\end{aligned} \tag{5.19}$$

Similarly, again by Lemma 3.4,

$$\begin{aligned}K_2 &\le \|\Gamma_1\|_{L^2((\partial\Omega)_\varepsilon)}\|\sigma_0\|_{L^2(\Omega_\varepsilon)}\\ &\le C(\sqrt{\varepsilon}\|\chi\|_{L^2(Y)}\|\widetilde{u}_0\|_{H^2(\mathbb{R}^d)})(\sqrt{\varepsilon}\|\sigma_0\|_{H^1(\Omega)})\\ &\le C\varepsilon\|u_0\|_{H^2(\Omega)}\|H\|_{L^2(\Omega)},\end{aligned} \tag{5.20}$$

and

$$K_3 \le \|\Gamma_2\|_{L^2(\Omega)}\|\sigma_0\|_{L^2(\Omega)} \le C\varepsilon\|u_0\|_{H^2(\Omega)}\|H\|_{L^2(\Omega)}. \tag{5.21}$$

By combining (5.19)-(5.21), we obtain

$$\left|\int_\Omega H\cdot\eta_0\right| \le C\varepsilon\|u_0\|_{H^2(\Omega)}\|H\|_{L^2(\Omega)},$$

which, by duality, leads to

$$\|\eta_0\|_{L^2(\Omega)} \leq C\varepsilon \|u_0\|_{H^2(\Omega)}. \tag{5.22}$$

Hence we have proved that

$$\|w_\varepsilon\|_{L^2(\Omega)} \leq \|\eta_\varepsilon - \eta_0\|_{L^2(\Omega)} + \|\eta_0\|_{L^2(\Omega)} + \|\phi_\varepsilon\|_{L^2(\Omega)} \leq C\varepsilon \|u_0\|_{H^2(\Omega)}. \tag{5.23}$$

The proof is finished. □

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October 13, 2015